MATH 6021 Lecture 9 11/9/2020

A brief introduction to varifolds

(Ref: L. Simon's book on GMT
Allard ~ '60s)

currents VS varifolds

Currents	Varifolds
• k-currents dual to k-forms	✓ • k-varifold dual to fcn on k-Grassmannian
• generalized <u>oriented</u> k-submfd ⇓	✓ • generalized <u>un-oriented</u> k-submfd
• notion of boundary ∂	• notion of 1^{st} variation ⇝ "stationary varifold"
• cptness result	✓ • cptness result
* • mass $\mathbb{M}$ is lower semi-cts	* • mass is continuous ✓

Recall: The k-Grassmannian in $\mathbb{R}^n$ is

$$Gr(k,n) := \{ V^k \subseteq \mathbb{R}^n : k\text{-dim'l (un-oriented) subspace} \}$$

↑ compact

(E.g.) $Gr(1,n) = \mathbb{RP}^{n-1}$)

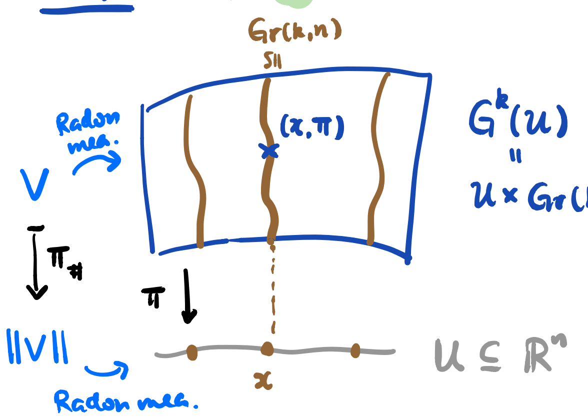
For any open set $\mathcal{U} \subseteq \mathbb{R}^n$, denote

$$G^k(\mathcal{U}) := \mathcal{U} \times Gr(k,n)$$

k-Grassmannian bundle over $\mathcal{U} \subseteq \mathbb{R}^n$.

Remark: Also works in manifolds, one way to define it using isometric embedding $M^n \hookrightarrow \mathbb{R}^N$.

Def$^{\underline{n}}$: A k-varifold V in $\mathcal{U} \subseteq \mathbb{R}^n$ is a Radon measure on $G^k(\mathcal{U})$.



Motivation: $\Sigma^k \subseteq \mathcal{U}$ k-submfd smooth

$x \in \Sigma \rightsquigarrow T_x\Sigma \in Gr(k,n)$

$\Rightarrow V \text{ "=" } \{ (x, T_x\Sigma) : x \in \Sigma \}$

Advantage: Can talk about 1^{st} variation!

Some related notions of varifolds:

- $V_i \longrightarrow V$ as varifolds if they converges as Radon measures, i.e.

$$\forall \varphi \in C_c(G^k(\mathcal{U})), \quad \int_{G^k(\mathcal{U})} \varphi(x,\pi)\, dV_i(x,\pi) \longrightarrow \int_{G^k(\mathcal{U})} \varphi(x,\pi)\, dV(x,\pi)$$

- $\exists$ Radon measure $\|V\|$, called the **weight of V**, on $\mathcal{U}$ s.t.

$\boxed{\|V\| = \pi_\# V}$, i.e. $\forall f \in C_c(\mathcal{U})$.

$$\int_{\mathcal{U}} f(x)\, d\|V\|(x) = \int_{G^k(\mathcal{U})} f(x)\, dV(x,\pi)$$

- $\operatorname{supp}(\|V\|)$ = smallest closed set outside which $\|V\|$ vanishes identically

- **mass** of V in $\mathcal{U}$ $:= \|V\|(\mathcal{U})$

Fact 1: Mass of varifolds is cts w.r.t the varifold topology.

Remark: Given a smooth embedded k-submfd $\Sigma \subseteq \mathcal{U}$, we can associate to it a k-varifold $|\Sigma|$ as follow:

$\forall \varphi \in C_c(G^k(\mathcal{U}))$,

$$\int_{G^k(\mathcal{U})} \varphi(x,\pi)\, d|\Sigma|(x,\pi) := \int_{\Sigma} \varphi(x, T_x\Sigma)\, d\mathcal{H}^k(x)$$

even allowing multiplicities:

$$\underbrace{V = \sum_i n_i |\Sigma_i|}_{\text{"integral varifolds"}}$$

where $n_i \in \mathbb{N}$, $\Sigma_i \subseteq \mathcal{U}$ smooth embedded k-submfd.

"integral varifolds" ← has good compactness properties.

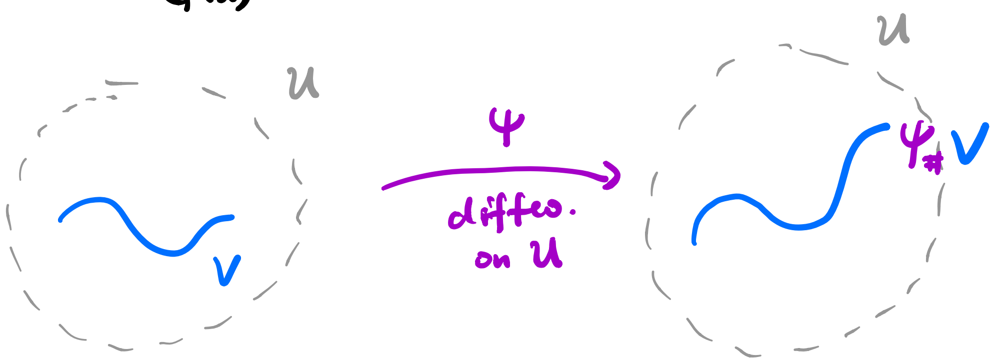
1^{st} variation of varifold

Let V be a k-varifold in $\mathcal{U} \subseteq \mathbb{R}^n$,

$\psi : \mathcal{U} \to \mathcal{U}$ be a diffeomorphism on $\mathcal{U}$.

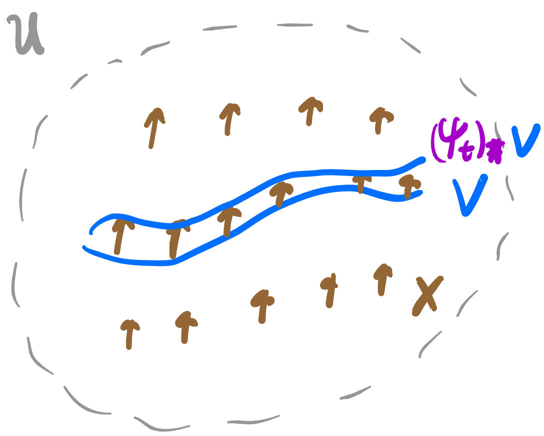
We can define the pushforward varifold $\psi_\# V$, which is a k-varifold on $\mathcal{U}$, as follows: $\forall \varphi \in C_c(G^k(\mathcal{U}))$.

$$\int_{G^k(\mathcal{U})} \varphi(y,\sigma)\, d(\psi_\# V)(y,\sigma) := \int_{G^k(\mathcal{U})} \varphi(\psi(x), (d\psi)_*(\pi)) \overbrace{J\psi(x,\pi)}^{\text{Jacobian}}\, dV(x,\pi)$$



Suppose X is a cptly supported vector field in $\mathcal{U} \subseteq \mathbb{R}^n$ and it generates $\{\psi_t\}_{t\in\mathbb{R}}$, 1-parameter family of diffeo. of $\mathcal{U}$

s.t $\psi_0 = id_{\mathcal{U}}$ and $\frac{\partial}{\partial t}\Big|_{t=0} \psi_t = X$



<u>Defn</u>: $\delta V(X) := \frac{d}{dt}\Big|_{t=0} \|(\psi_t)_\# V\|(\mathcal{U})$

$$= \int_{G^k(\mathcal{U})} \underbrace{\mathrm{div}_\pi X(x)}_{= \sum_{i=1}^{k} \langle D_{e_i} X, e_i \rangle (x)} \, dV(x,\pi)$$

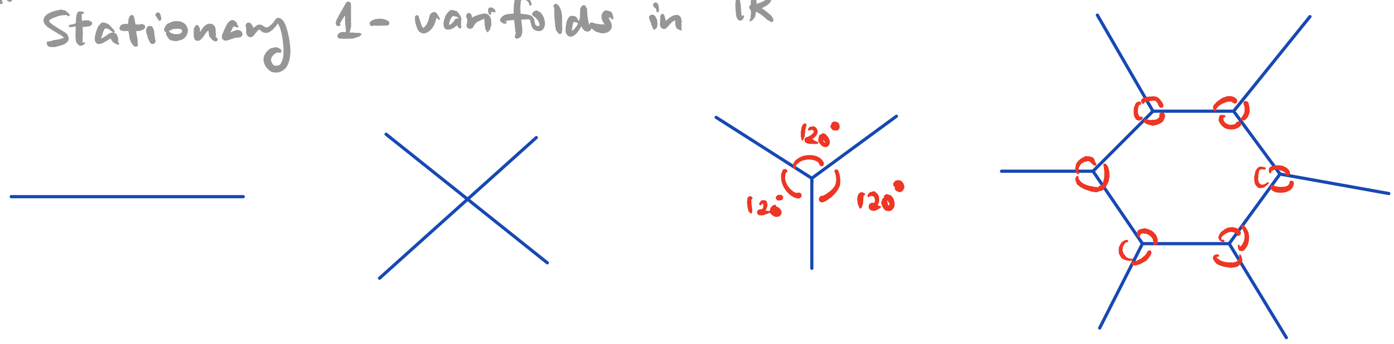
$\{e_i\}$ o.N.B. for π

1^{st} variation formula for varifolds.

Def$^{\underline{n}}$: V is stationary in $\mathcal{U}$ if $\delta V(X) = 0 \quad \forall X$.

Example: any k-submfd minimal is stationary.

" Stationary 1-varifolds in $\mathbb{R}^2$ "



Facts about stationary varifolds (c.f. L. Simon)

(i) Monotonicity Formula: $\dfrac{\|V\|(B_r(x))}{r^k} \nearrow$ in r

(ii) $\Theta(x, V) := \lim\limits_{r \searrow 0} \dfrac{\|V\|(B_r(x))}{\sigma_k r^k}$ well-defined.

[Idea: $\Theta = 1 \Rightarrow V$ is smooth]

(iii) Allard Regularity: " $\Theta \approx 1 \Rightarrow V$ is a $C^{1,\alpha}$ k-submfd. ".

(iv) Rectifiability Thm: V stationary. $\Theta(x, V) > 0$ for $\|V\|$-a.e x

$\Rightarrow V$ is rectifiable.

Summary: Θ contains useful information about regularity.

Continuous Min-Max Theory on 3-manifolds

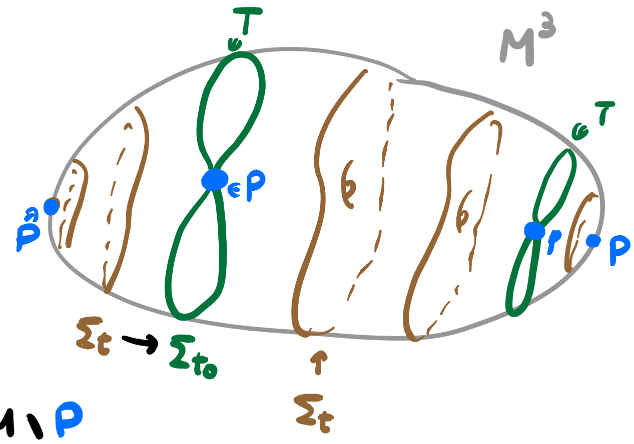
[Ref: L. Simon. & F. Smith '80s, Colding - De Lellis '03]

Setup: (M^3, g) closed Riem. mfd

$\mathcal{D}\mathrm{iff}_0 :=$ id_M component of $\mathcal{D}\mathrm{iff}(M)$.

$\underline{\text{Def}^{\underline{n}}}$: $\{\Sigma_t\}_{t\in[0,1]}$ is a **generalized family of surfaces**

if $\exists$ finite subsets $T \subseteq [0,1]$ & $P \subseteq M$ s.t.



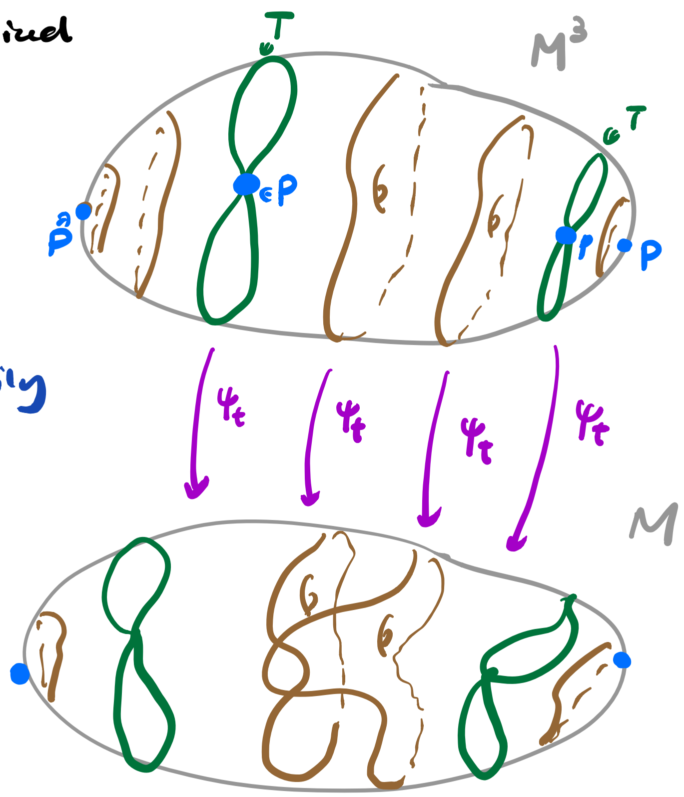
- $t \mapsto \mathcal{H}^2(\Sigma_t)$ cts
- $\Sigma_t \to \Sigma_{t_0}$ in "Hausdorff" topology as $t \to t_0$
- Σ_t is smooth embedded $\forall\, t \notin T$
- $\forall\, t \in T$, Σ_t is smooth embedded in $M \setminus P$

$\underline{\text{Fact}}$: Given $\{\Sigma_t\}$ as above, and

$$\Psi : [0,1] \times M \to M \quad \text{s.t.} \quad \Psi(t,\cdot) \in \mathrm{Diff}_0 \quad \forall\, t \in [0,1]$$

then $\{\Sigma_t' := \Psi(t, \Sigma_t)\}$ is another generalized family

We say a collection Λ of generalized family is **saturated** if it is closed under such operation.

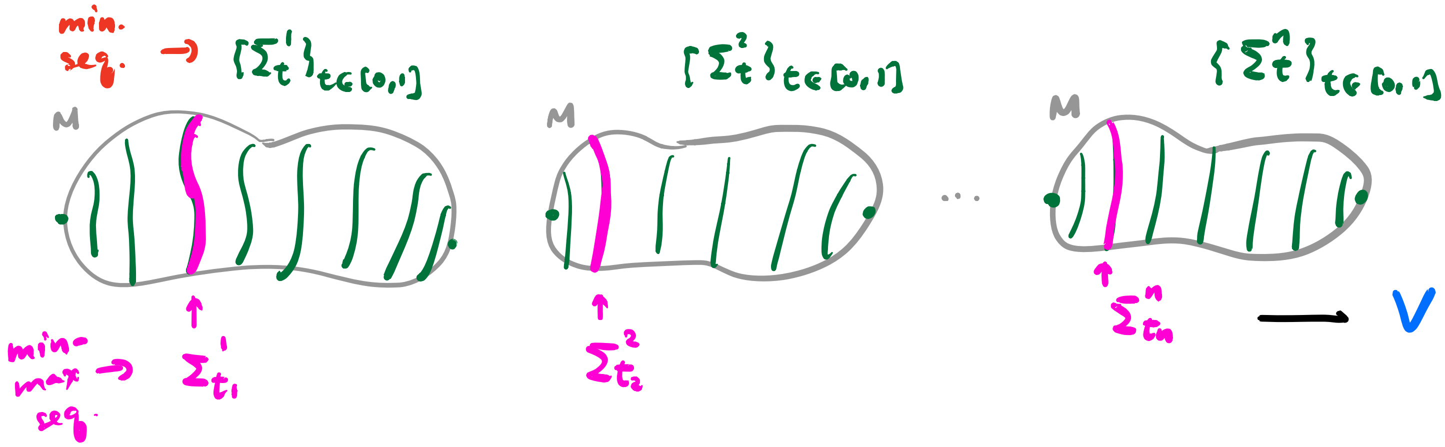


$\underline{\text{Def}^{\underline{n}}}$: Given such a saturated collection Λ of generalized family of surfaces, define

$$m_0(\Lambda) := \inf_{\{\Sigma_t\}\in\Lambda} \left(\max_{t\in[0,1]} \mathcal{H}^2(\Sigma_t) \right)$$

$\underline{\text{Terminology}}$:

- **minimizing seq.** $\{\Sigma_t^n\}_{n\in\mathbb{N}}$ of generalized family if $\max_{t\in[0,1]} \mathcal{H}^2(\Sigma_t^n) \to m_0$ as $n \to \infty$.
- **min-max seq.** $\{\Sigma_{t_n}^n\}_{n\in\mathbb{N}}$ of surfaces if $\mathcal{H}^2(\Sigma_{t_n}^n) \to m_0$ as $n \to \infty$



Min-Max Thm : $\exists$ min-max seq. $\{\Sigma_{t_n}^n\}_{n\in\mathbb{N}}$ of surfaces s.t.

(i) $\Sigma_{t_n}^n \longrightarrow V$ as varifolds

(ii) V is a stationary integral varifold and

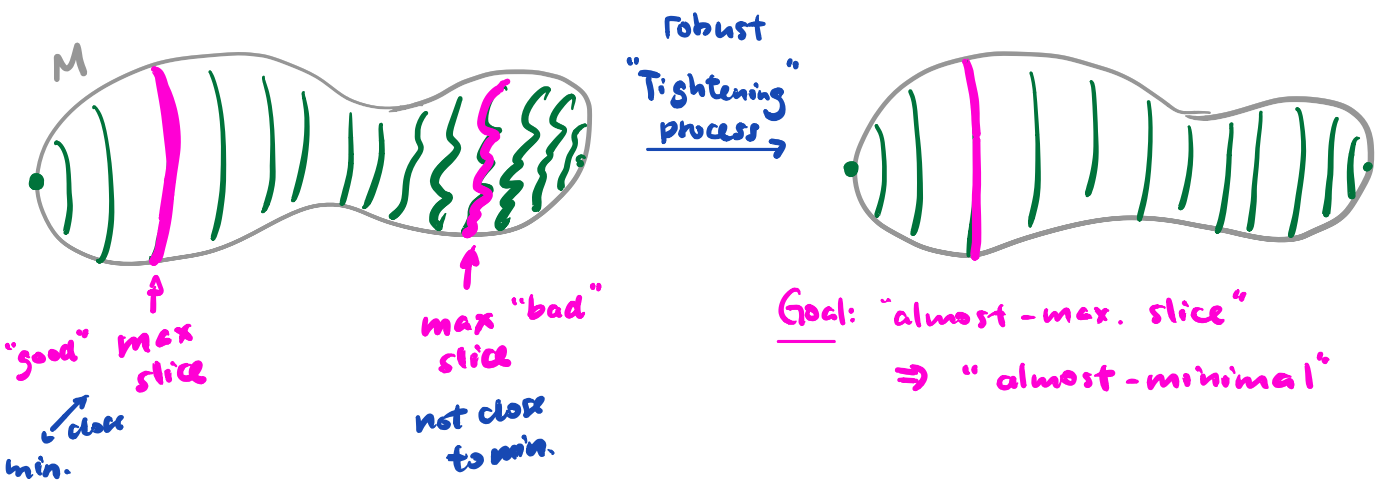
$$V = m_1 |T_1| + \cdots + m_q |T_q|$$

where $m_i \in \mathbb{N}$, $T_i \subseteq M$ smooth embedded min. surface.

"**Rough Ideas of the Proof**":

(i) "Existence of stationary varifolds" (works in any dim / co-dim)

Problem? Slices with area $\approx m_0$ may not be "close" to a min. surface.



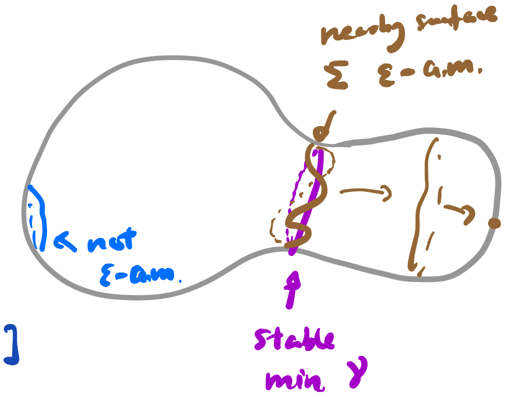
Goal: "almost-max. slice" ⇒ "almost-minimal"

Solution: $\exists$ min. seq. $\{\Sigma_t^n\} \subseteq \Lambda$ s.t. EVERY min-max seq. $\{\Sigma_{t_n}^n\}$ has subseq. converging weakly to a stationary varifold.

(ii) "Regularity"

The key variational argument is the "almost-minimizing property".

Def^n: Let $\varepsilon > 0$, small, and $\mathcal{U} \subseteq M$ open.



- A surface $\Sigma \subseteq M$ is ε-a.m. in $\mathcal{U}$ if $\nexists$ isotopy Ψ supported in $\mathcal{U}$ s.t.

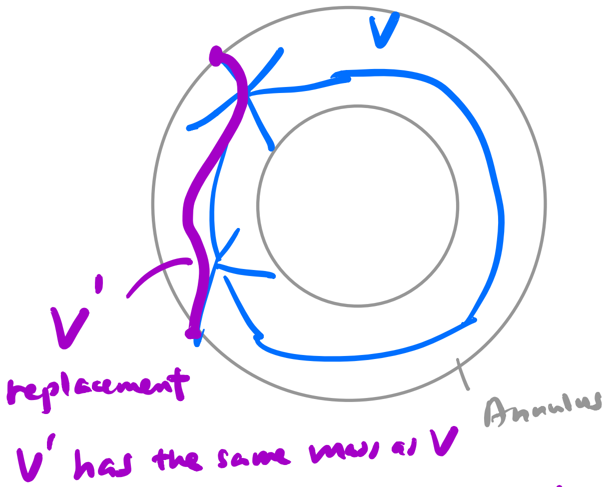
  (1) $\mathcal{H}^2(\Psi(t,\Sigma)) \leq \mathcal{H}^2(\Sigma) + \varepsilon/8 \quad \forall\, t \in [0,1]$

  (2) $\mathcal{H}^2(\Psi(1,\Sigma)) \leq \mathcal{H}^2(\Sigma) - \varepsilon$

- A seq $\{\Sigma^n\}$ is a.m. in $\mathcal{U}$ if each Σ^n is ε_n-a.m. in $\mathcal{U}$ with $\varepsilon_n \searrow 0$.

Prop: $\exists$ min-max seq. $\{\Sigma^n_{t_n}\}$ which a.m. in small annuli.

$$\text{a.m. property} \xRightarrow[\text{minimize in isotopy class}]{\text{Meeks-Simon-Yau '82}} \text{good replacement property} \xRightarrow[\text{regularity of stable min. surfaces.}]{\text{Schoen-Simon-Yau '75}} \text{smoothness of } V$$



- V' has the same mass as V
- V' in Ann is a smooth min surface

$\square$